

Synchronization learning of coupled chaotic maps

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We study the dynamics of an ensemble of globally coupled chaotic logistic maps under the action of a learning algorithm aimed at driving the system from incoherent collective evolution to a state of spontaneous full synchronization. Numerical calculations reveal a sharp transition between regimes of unsuccessful and successful learning as the algorithm stiffness grows. In the regime of successful learning, an optimal value of the stiffness is found for which the learning time is minimal.

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I. INTRODUCTION

Synchronization is a form of macroscopic evolution observed in a wide class of complex systems. Typically, it appears when the range of the interactions inside the system is of same order as the system size. Mechanical and electronic devices, as well as certain chemical reactions [1] are known to exhibit synchronized dynamics. In the realm of biology, among many other instances [2], synchronization appears at the cellular level in neural networks [3] and in heart tissues [4]. Animal populations show also complex forms of synchronous behavior, the most spectacular example being probably the synchronous flashing of certain fireflies [5]. The spontaneous occurrence of synchronization in biological systems suggests that this form of collective behavior develops as a consequence of evolutionary selection or of some kind of adaptive learning. In this paper we explore a model where a set of coupled chaotic elements is added with a feedback learning process targeting a collective synchronized state. We find that, under suitable conditions, the system evolves from completely incoherent behavior to a state of full synchronization.

Collective behavior under the effect of long-ranged interactions can be modelled by means of ensembles of globally coupled dynamical systems. Introduced by Kaneko a decade ago [6], globally coupled logistic maps have proven to be an appropriate paradigm for such kind

of emerging evolution. The system consists of N identical mappings whose individual dynamics, in the absence of interactions, is given by $x(t+1) = F[x(t)]$ with $F(x) = rx(1-x)$. The individual dynamics are coupled according to

$$x_i(t+1) = (1-\epsilon)F[x_i(t)] + \frac{\epsilon}{N} \sum_{j=1}^N F[x_j(t)], \quad (1)$$

($i = 1, \dots, N$), where $\epsilon \in (0, 1)$ is the coupling intensity.

The system defined by Eqs. (1) features a variety of collective behaviors in the space spanned by the nonlinear parameter r and the coupling constant ϵ [6]. At large values of ϵ the ensemble is in a *coherence phase*, where all the elements tend asymptotically to exactly the same trajectory $x(t)$. For sufficiently long times, thus, the system is fully synchronized. Note that the trajectory of the synchronized ensemble is governed by the dynamics of a single element. At low ϵ , when coupling is weak, a *turbulence phase* is observed, where the evolution is completely unsynchronized. At intermediate values of the coupling intensity a *clustering phase* with different groups of mutually synchronized elements appears.

Bearing in mind the role of synchronization as a collective acquired behavior of biological systems, we incorporate to model (1) an additional evolutionary mechanism. Concretely, each element is allowed to vary its coupling constant in time, so that the effect of the collective evolution on its dynamics—defined by the last term in the right-hand side of (1)—is modified according to a given criterion. Thus, the system may be able to learn to perform a specific collective task, in particular to evolve towards a coherent synchronized state. We focus the attention in the properties of the learning process, and find that a regime of successful learning exists for a sufficiently stiff algorithm. Within this regime, the time necessary to achieve synchronization is determined by the stiffness.

II. MODELING

We consider a variation of model (1) where the coupling constants depend on time and may be different for each element, namely,

$$x_i(t+1) = [1 - \epsilon_i(t)]F[x_i(t)] + \frac{\epsilon_i(t)}{N} \sum_{j=1}^N F[x_j(t)]. \quad (2)$$

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Inhomogeneous global coupling in ensembles of logistic maps with time-independent coupling constants has been considered in previous work [7,8]. It can be shown that the system exhibits full synchronization if and only if all the coupling constants ϵ_i are larger than a certain value ϵ_c . This value turns out to coincide with the critical point for the onset of full synchronization in homogeneous ensembles, and can be given in terms of the Lyapunov exponent λ of a single map as $\epsilon_c = 1 - \exp(-\lambda)$ [9]. In particular, for a nonlinear parameter $r = 4$ we have $\lambda = \ln 2$ and therefore $\epsilon_c = 1/2$.

Time variation of coupling intensities as a form of adaptive behavior has been considered in globally coupled maps by analogy with synaptic evolution in neural networks [7], and in models of asymmetric imitative dynamics for two-element systems [10]. Here, we consider a learning algorithm based on a comparison of the instantaneous state $x_i(t)$ of each element with a global property of the ensemble, namely the instantaneous average state $\langle x(t) \rangle = N^{-1} \sum_j x_j(t)$. If the distance from the individual state to the average is larger than a certain threshold u the learning process acts, and the coupling constant of the element is changed to a new value, chosen at random from a uniform distribution in $(0, 1)$. Otherwise, ϵ_i remains unchanged. Explicitly,

$$\epsilon_i(t+1) = \begin{cases} \xi_i(t+1) & \text{if } |x_i(t+1) - \langle x(t+1) \rangle| > u \\ \epsilon_i(t) & \text{otherwise,} \end{cases} \quad (3)$$

where ξ_i is a random number with uniform distribution in $(0, 1)$. Note that the threshold u can be interpreted as an inverse measure of the stiffness of learning. The evolution proceeds according to the following dynamical rules. First, the state $x_i(t)$ of every element is updated to $x_i(t+1)$ applying map (2). The average state $\langle x(t+1) \rangle$ is then calculated. Finally, the learning algorithm (3) is applied to every element. This procedure is successively iterated, so that each evolution step consists of two sub-steps where coupled dynamics and learning act sequentially. Both processes are applied synchronously to the whole system.

This is a form of stochastic unsupervised learning [11] where the whole ensemble is expected to selforganize into a coherent state where the orbit of every element coincides with the average trajectory. The algorithm can be interpreted as an adaptive control mechanism [12]. Recently, control techniques have been proposed to drive both low-dimensional and extended dynamical systems towards a prescribed state, such as a particular spatiotemporal pattern [13,14]. The present variant is inspired in arguments of biological plausibility. First, the target of learning is not a specific dynamical state, but a wide class defined by a collective property, namely synchronization. This class includes not only infinitely many synchronized orbits but also a variety of different config-

urations of the set of couplings ϵ_i . Second, the learning algorithm acts at the individual level. That is, at each time step the criterion (3) is applied to each element. According to its individual state a modification is introduced to its coupling. The collective state achieved through learning emerges thus as a consequence of individual evolution. Finally, the modification applied to the coupling intensities is random and unbiased. No hints are given on the desired values of the coupling intensities, which must be adaptively found by the system through iterations of trials and errors. Successive values of ϵ_i are completely uncorrelated.

In the following section, we report results of extensive numerical realizations of the above model. They correspond to an ensemble of logistic maps with $r = 4$, i.e. at the fully developed chaotic regime. The system is investigated as a function of the threshold u . We detect a sharp transition at $u = 0.5$, between a regime of successful learning and a regime where learning fails. The origin of this transition is identified by studying the intermittent dynamics just before the state of full synchronization is reached. We define a parameter that measures the performance of learning and find an optimal value of u for which learning is fastest.

Fig. 1 - Moyano, Abramson & Zanette - Synchronization learning...

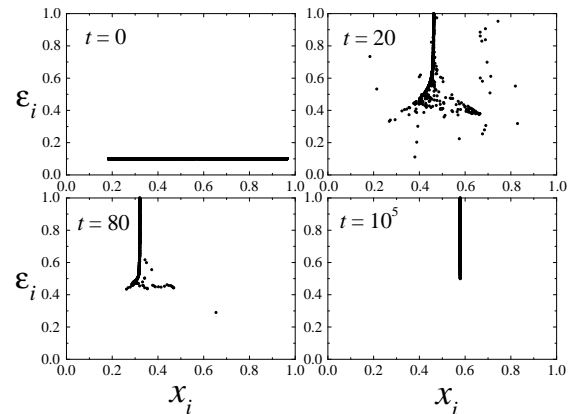


FIG. 1. Snapshots of an ensemble of $N = 1000$ elements on the (x_i, ϵ_i) -plane at four times: $t = 0$, $t = 20$, $t = 80$, and $t = 10^5$. The learning threshold is $u = 0.2$ and learning is successful.

III. RESULTS

The numerical results reported in this section correspond to systems with $N = 10^3$ elements. The initial states $x_i(0)$ are distributed at random, with uniform density, in the interval $(0, 1)$. All the coupling constants have initially the same value, $\epsilon_i(0) = \epsilon_0 = 0.1$. For this coupling intensity and a nonlinear parameter $r = 4$, the initial state of the system is well within the turbulence phase [6]. During a first stage, the ensemble is left to evolve 10^3 steps without applying the learning dynamics,

such that the individual states x_i adopt the characteristic distribution of an incoherent state. After this, time is reset to zero, learning is switched on, and the states and coupling constants of all elements are recorded during the following 10^4 to 10^6 steps.

A suitable way of representing the instantaneous state of the system is a plot where each element is shown as a dot in the plane spanned by the individual state x_i and the coupling constant ϵ_i [7,8]. Figure 1 shows four such snapshots for a single system at different times, corresponding to a threshold $u = 0.2$. At the initial time $t = 0$ all elements have the same coupling constant ϵ_0 , and form an incoherent, extended cloud in x_i . At subsequent times, $t = 20$ and $t = 80$, the effects of learning are clearly visible. The elements that have migrated to larger values of ϵ_i form now a rather compact cluster though, for $t = 20$, many elements with relatively large coupling constants are still far from the main cluster. For $t = 80$, almost all the elements have got coupling constants above the critical value $\epsilon_c = 0.5$. The remaining elements form a small cloud just below ϵ_c , and approximately follow the motion of the main cluster. Finally, for $t = 10^5$ all the elements have $\epsilon_i > 0.5$ and the same value of x_i . For this value of u , learning has been successful and the ensemble has become fully synchronized.

Fig. 2 - Moyano, Abramson & Zanette - Synchronization learning...

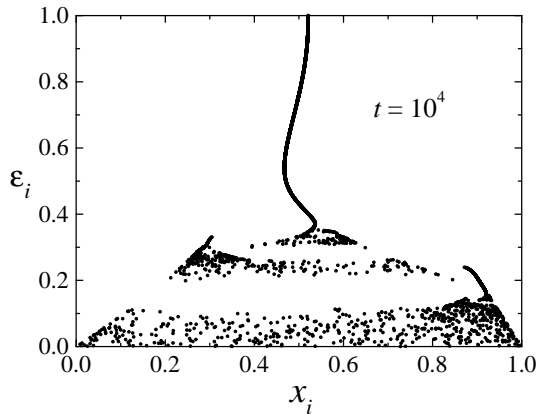


FIG. 2. Snapshot of an ensemble of $N = 1000$ elements on the (x_i, ϵ_i) -plane with $u = 0.6$ at $t = 10^4$. At this value of the threshold u learning fails to drive the system to full synchronization.

We show in Fig. 2 a situation where, on the other hand, learning is not able to lead the system to the synchronization phase. Here $u = 0.6$ and the system has been left to evolve for $t = 10^4$ steps. After an initial redistribution of the coupling constants, the ensemble appears to have reached a stationary state with a complex organization in the (x_i, ϵ_i) -plane (compare with analogous distributions reported in [7,8]) but with no traces of synchronization. According to our simulations, this state is preserved at longer times, of the order of 10^6 steps.

These preliminary results suggest that some kind of transition occurs at an intermediate value u_c of the threshold, separating two regions where learning is respectively successful and unsuccessful. This transition is characterized in the following.

A. Learning transition

As a global measure of the collective state of the ensemble during its evolution we have chosen the mean dispersion of the individual states x_i , namely, $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$. For the fully synchronized state, $\sigma_x = 0$. We have studied the evolution of $\sigma_x(t)$ for several values of the threshold u , and found that for $u > 0.5$ the dispersion asymptotically approaches a finite value. The average of this asymptotic value over different realizations for a fixed threshold is practically independent of u , $\bar{\sigma}_x \approx 0.25$. This is the dispersion that corresponds to the state shown in Fig. 2.

For $u < 0.5$, on the other hand, we have always found that $\sigma_x \rightarrow 0$ for sufficiently long times. We stress, however, that the typical times associated with this evolution depend strongly on the threshold, as shown in detail later. In any case, for such values of u , the ensemble approaches asymptotically the state of full synchronization. Thus, the critical threshold $u_c \approx 0.5$ is the boundary between the zone of successful learning ($u < u_c$) and the zone where the system fails to learn how to synchronize ($u > u_c$). As discussed, the average asymptotic dispersion $\bar{\sigma}_x$ is sharply discontinuous at u_c .

Fig. 3 - Moyano, Abramson & Zanette - Synchronization learning...

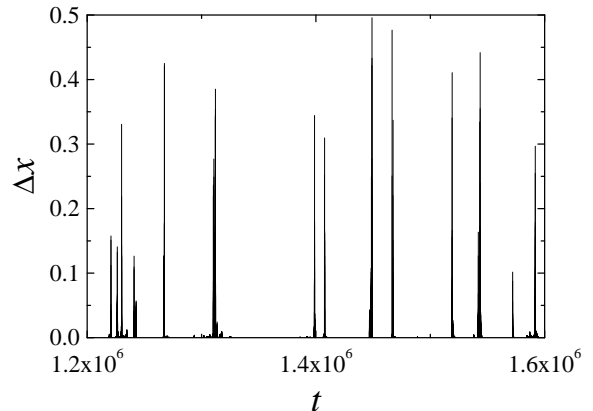


FIG. 3. Intermittent evolution of a single element that is prevented to evolve, with coupling constant $\epsilon_i = 0.4997$. The plot shows the absolute difference between the state of the element and the average state of the system. The learning threshold is $u = 0.2$.

Actually, this abrupt transition in the performance of learning can be explained by studying the dynamics of our system in the intermediate stages of evolution, when a considerable fraction of the ensemble already defines a

compact cluster whereas the remaining elements, whose coupling constants are just below $\epsilon_c = 0.5$, form the small cloud depicted in Fig. 1 for $t = 80$. In this situation, the system is in the threshold of a bifurcation where the stable state of full synchronization appears. Indeed, it would suffice to slightly change the coupling constants of the elements in the cloud to values above ϵ_c in order to create an attractor corresponding to the synchronized state. In the threshold of such bifurcation the system is expected to display intermittent evolution [15]. We have in fact verified in the numerical simulations that the dynamics of each element in the cloud exhibits two distinct regimes. Most of the time, the element is found in a “laminar” regime, where its state is practically equal to that of the main cluster. Occasionally, however, the evolution exhibits “turbulent” bursts during which the element performs short excursions far away from the cluster. In order to illustrate this behavior, we have recorded the evolution of a single element in the ensemble, whose coupling constant is initially fixed at $\epsilon_i = 0.4997$. Learning is subsequently prevented for this special element, in such a way that its coupling constant remains fixed as time elapses. For long times, $t \sim 10^6$, the coupling constants of all the other elements are found above ϵ_c . Intermittency is apparent in Fig. 3, where the difference $\Delta x = |x_i - \langle x \rangle|$ has been plotted for the special element as a function of time. Here, the average state $\langle x \rangle$ is essentially determined by the position of the main cluster.

As expected, the intervals of “laminar” behavior are found to grow in length as the coupling constant of the element under study approaches ϵ_c . Conversely, the frequency of “turbulent” bursts decreases. Note that, for the elements in the cloud, learning is possible during these bursts only. In fact, according to (3), learning acts when the difference between the individual state and the average state is large enough. Thus, the closer the coupling constant is to ϵ_c , the later an element undergoes a learning step.

The dynamics of a single element in the cloud below ϵ_c can be well approximated as follows. We disregard the effect of the remaining elements in the cloud and suppose that the interaction of the element under study with the ensemble occurs only through the main cluster. Conversely, we suppose that the main cluster contains essentially all the elements of the system in a synchronized state $x_0(t)$, and that is not affected by the dynamics of the cloud. Within these assumptions, the state $x(t)$ of the element under study, whose coupling constant is $\epsilon \lesssim \epsilon_c$, evolves according to

$$x(t+1) = (1 - \epsilon)F[x(t)] + \epsilon F[x_0(t)]. \quad (4)$$

Meanwhile, the state of the main cluster obeys the dynamics of a single independent map, $x_0(t+1) = F[x_0(t)]$. The analytical study of the intermittent evolution of $x(t)$ from Eq. (4) may be difficult. However, we can easily

find bounds for the excursions of $x(t)$ during the bursts. Note in fact that we can write

$$\begin{aligned} \Delta x(t+1) &= |x(t+1) - x_0(t+1)| \\ &= (1 - \epsilon)|F[x(t)] - x_0(t+1)|. \end{aligned} \quad (5)$$

Putting $F(x) = 4x(1 - x)$ and taking into account that at any time both $x(t)$ and $x_0(t)$ are in the interval $(0, 1)$, we find $\Delta x < 1 - \epsilon$ which, for $\epsilon \rightarrow \epsilon_c$ reduces to

$$\Delta x < 1/2. \quad (6)$$

This bound is clearly seen in Fig. 3. We conclude that the distance from the main cluster to an element in the cloud during a burst cannot be larger than $1/2$. For such an element, consequently, learning will occur during a sufficiently ample burst only if $u < 0.5$. If $u > 0.5$, the learning algorithm will never be applied and the system will not reach the fully synchronized state. This fixes the learning transition at $u_c = 0.5$.

B. Learning times

In order to give a more detailed description of the evolution of our system under the action of learning, we focus now the attention on the time needed to approach the synchronization state. To define such a time, we study first the fraction $n(t)$ of elements that, at a certain moment, have their coupling constants below ϵ_c . We have $n(0) = 1$ and, for $u < u_c$, $n(t) \rightarrow 0$ as $t \rightarrow \infty$.

Fig. 4 - Moyano, Abramson & Zanette - Synchronization learning...

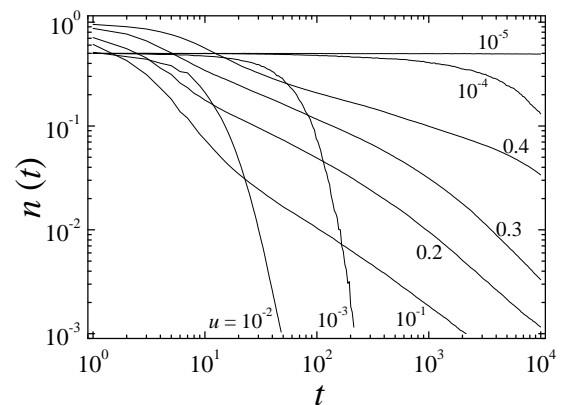


FIG. 4. Time evolution of the fraction $n(t)$ of elements with $\epsilon_i(t) < 0.5$ in ensembles of $N = 1000$ elements. Different curves correspond to systems with different values of u , as shown.

Figure 4 shows the decay of $n(t)$ as a function of time, for several values of the threshold $u < u_c$. Each curve is the average of several hundred realizations. We see that the decay is monotonous, though for some values of u the evolution is extremely slow. This happens near

$u = 0$ and $u = 0.5$, whereas at intermediate values of the threshold the decay is faster.

It is apparent from Fig. 4 that, for different values of the threshold, the functional form of $n(t)$ is not uniform. In order to define a characteristic time associated with learning, then, we fix a reference level n_0 and measure the time $T(u)$ needed for $n(t)$ to reach that level for each value of u . A plausible value for the reference level is $n_0 \sim N^{-1}$, which indicates that for $t > T(u)$ only a few elements remain in the unsynchronized cloud. In our simulations, the learning time $T(u)$ has been determined taking $n_0 = N^{-1} = 10^{-3}$. It thus corresponds to the time taken by all elements but one to migrate to the main cluster. The learning time as a function of the threshold u is shown in Fig. 5. Each dot stands for the average of several hundred realizations. There is a clear minimum at $u \approx 0.01$ showing that, as for the performance of learning, there is an optimal choice for the threshold u . Note that, as far as $n_0 \sim N^{-1}$, the position of the minimum does not depend on the reference level.

Fig. 5 - Moyano, Abramson & Zanette - Synchronization learning...

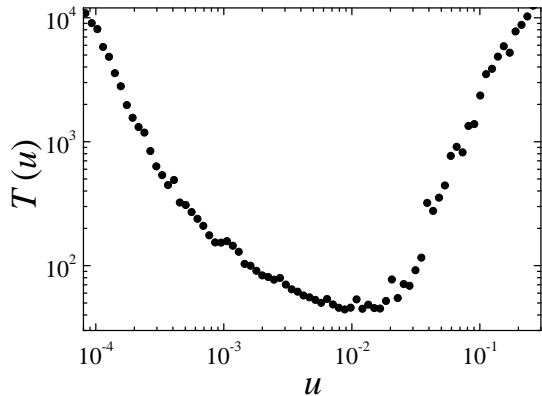


FIG. 5. Average learning time $T(u)$ as a function the threshold u in ensembles of $N = 1000$ elements.

The presence of an optimal threshold for our learning algorithm can be explained as follows. For large values of the threshold, $u \lesssim 0.5$, the formation of the small cloud at $\epsilon \lesssim 0.5$ and the ensuing appearance of intermittent evolution occur relatively fast. Once this situation has been established, however, the elements in the cloud spend very long times in the “laminar” regime and in unsuccessful bursts, whose amplitude is not enough to drive the elements beyond the threshold. Successful bursts, where $\Delta x > u$ [see Eq. (5)], become in fact increasingly rare as $u \rightarrow u_c$ and, consequently, the learning time is expected to diverge in such limit. At the other end, $u \approx 0$, the threshold is very narrow and the elements keep changing their coupling constants for long times. They need many attempts to approach the average behavior. Therefore, even the initial stage during which the main cluster is formed lasts asymptotically large times as $u \rightarrow 0$. Since

the learning time $T(u)$ should diverge both at $u = 0$ and at $u = u_c$, it must reach (at least) a minimum for an intermediate value of the threshold, as fully confirmed by our numerical results.

IV. SUMMARY AND CONCLUSION

We have studied an ensemble of globally coupled chaotic maps able to change their individual couplings in order to evolve from an incoherent collective dynamics to a completely synchronized state. The learning procedure is implemented by means of a stochastic unsupervised algorithm characterized by a single parameter u than measures the stiffness of learning. Our numerical results show that the emergence of synchronization is only possible for a specific range of the parameter u . In fact, a sharp transition at $u_c = 0.5$ has been found, separating a regime of successful learning ($u < u_c$) from a regime where the algorithm fails to drive the system to full synchronization ($u > u_c$). In the zone of successful learning, in turn, the time needed to reach a certain level of the learning process has been shown to strongly depend on u . In particular, we have found that there is an optimal value, $u \approx 0.01$, for which the learning time is minimum, i.e. learning is fastest.

The sharp transition between the regimes of successful and unsuccessful learning can be explained taking into account the intermittent evolution observed just before the state of full synchronization has been reached. This kind of evolution is in fact typical at the threshold of a synchronization transition [16]. In this regime it is possible to formulate an approximate dynamical description that yields the bounds for the amplitude of intermittent bursts, during which the elements are subject to the learning process. These bounds define in turn the maximal value u_c for which learning is possible. As for the optimal value of u , we have argued that both for $u \rightarrow u_c$ and for $u \rightarrow 0$ the learning time is expected to diverge, so that at least one minimum should be found for intermediate values. The existence of an optimal value for the learning stiffness should be a generic property of a large class of learning algorithms. This point has already been discussed to some extent in connection with several training algorithms for neural networks [11]. Indeed, every teacher should know that there is an optimal “pressure” to be applied on the average student to obtain the best and fastest results in learning.

Several generalizations to the present model can be foreseen, attempting to describe other situations found in real systems. Our learning algorithm, in fact, is based on the comparison of the individual state of each element with a global quantity, namely, the average state over the ensemble. This could be replaced by a sort of “local” criterion, where the comparison takes place between pairs

[10] or small groups of elements. Moreover, coupling constants could be subject to smoother changes, representing a smarter learning process, instead of the trial-and-error method used here. In the line of some training algorithms for neural networks and of optimization schemes, a variation of the present model would consist in allowing the learning stiffness to change with time. A suitably controlled temporal variation for u could in fact result in a substantial decrease of the learning times. Learning itself could be fully replaced by an evolutionary mechanism, in the spirit of genetic algorithms. In this case, unsuccessful elements should be eliminated and replaced by slightly modified copies of successful elements, which are the effects expected from natural selection and mutation, respectively. Finally, the individual dynamics of the coupled elements admits to be varied within an ample class of behaviors, including discrete and continuous evolution. The present work, in summary, is a first step in the study of a wide spectrum of problems, of interest from the viewpoint of biology, optimization techniques, and artificial intelligence.

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